equations that follow by assuming a circular orbit (so that $\dot{R} = \ddot{R} = 0$). Then by adding to the constant energy E a quantity corresponding to its value in the basic orbit, $-G/2R$, a function $V(\rho,\dot{\rho})$ is obtained in the form

$$V(\rho,\dot{\rho}) = E + \frac{G}{2R} = \frac{\dot{\rho}^2}{2} + \frac{G}{R^3}\frac{\rho^2}{2}$$

when all third and higher powers of ρ and $\dot{\rho}$ are neglected.

The function V so defined has the following properties: 1) $V(\rho,\dot{\rho})$ and its first partial derivatives are continuous in a region that includes the point $\rho = \dot{\rho} = 0$; 2) $V(0, 0) = 0$, an isolated minimum; and 3) $V > 0$ in a region that excludes only the point (0, 0), and the function is positive definite. Any function that satisfies these conditions and, in addition, is such that $\dot{V} \leqq 0$ is called a Liapunov function. It is verified easily that in this case $\dot{V} = 0$, identically, in fact, by noticing that

$$\dot{V} = \dot{\rho}\cdot[\ddot{\rho} + (G/R^3)\rho] = 0$$

the term in brackets being exactly the derivative of the (constant) energy of the motion. Under these circumstances, Chetayev's proofs are applicable and establish that the motion is stable.[1] It is noted further that, because $-\dot{V}$ is *not* positive definite, the motion also is not asymptotically stable (that is, the disturbance does not ultimately decay and restore the "basic" orbit).

Analogous results completely are obtained also for orbits of nonzero eccentricity with these relatively minor differences of detail: that the Liapunov function contains a greater number of terms corresponding to the fact that the eccentricity is nonzero, and that the angular velocity no longer is suppressed so easily as in the present case by introduction of the constant h. Even when librational interactions with orbital disturbances raise the order of the differential equation system, the same procedure exactly avails with no essential complication or loss of rigor.

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Approximate Longitudinal Dynamics of a Lifting Orbital Vehicle

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IN Ref. 1 an analysis is made of the dynamic longitudinal stability of a vehicle on an orbital path. Analysis of the mode shapes showed that, for two of the modes, the phugoid and the arrow, the angle of attack varied only slightly. It is of interest to compare the numerical results of Ref. 1 with those obtained from an analysis in which the a priori assumption of constant angle of attack is made.

The derivation of the equations for constant angle of attack is straightforward, with those given in Ref. 2 being applicable to this case also. Comparisons between the results from the two methods are presented in Figs. 1–3. The approximate results are within about 30% of the exact results for the phugoid period, with somewhat more deviation obtained for the phugoid damping. These results are about what should be expected, based upon the known accuracy of the

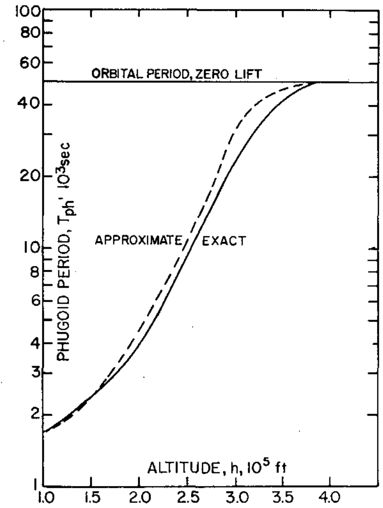

Fig. 1 Comparison of results, phugoid period

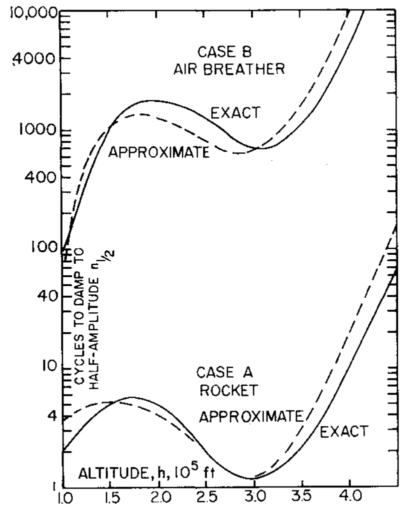

Fig. 2 Comparison of results, phugoid damping

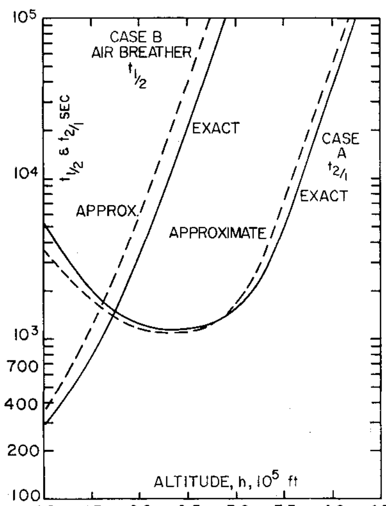

Fig. 3 Comparison of results, "arrow" mode

approximation for the classical phugoid. The characteristics of the arrow mode are predicted within a factor of about 2. Although the numerical accuracy is not particularly high, the results do give fairly good comparison over a very wide range of conditions. The results, therefore, should be adequate for preliminary estimates.

The details are given in Ref. 3, along with generalized charts based upon the same approximation.

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Optimum Deboost Altitude for Specified Atmospheric Entry Angle

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Nomenclature

h = altitude
r = radius
V = velocity
ΔV = velocity increment
β = retrofire orientation angle
γ = elevation angle above local horizontal

Subscripts

0 = conditions at planet radius
1 = conditions at initial altitude
c = circular
E = conditions at entry altitude

Introduction

Techniques for the controlled recovery of a re-entry vehicle by impulsive deboost are well known. The general problem has been discussed by Low[1] for near-earth orbits through a linearization technique. Detra, Riddell, and Rose[2] have examined the problem with respect to some of the limitations imposed by weight and accuracy requirements. Galman[3] and Low[1] have discussed maximizing the atmospheric entry angle for a given retrorocket velocity decrement.

The purpose of this note is to show that there is a minimum (retrofire) impulse with respect to altitude for a given entry angle. This result suggests that a mission altitude could be selected for minimum deboost requirement. It is also shown that minimizing the retrofire impulse with respect to retrofire angle is equivalent to maximizing the atmospheric entry angle (the problem investigated by Low[1] and by Galman[3]).

Analysis

Consider a vehicle initially in a circular orbit as shown in Fig. 1. From the conservation of energy and angular momentum,

$$V_1^2 - 2V_{c_1}^2 = V_E^2 - 2V_{c_1}^2(r_1/r_E) \tag{1}$$

$$r_1 V_1 \cos\gamma_1 = r_E V_E \cos\gamma_E \tag{2}$$

Also, from geometric considerations,

$$V_1^2 = V_{c_1}^2 + (\Delta V)^2 - 2V_{c_1}\Delta V \cos\beta \tag{3}$$

$$V_1 \cos\gamma_1 = V_{c_1} - \Delta V \cos\beta \tag{4a}$$

or the equivalent

$$V_1/\sin\beta = \Delta V/\sin\gamma_1 \tag{4b}$$

It should be noted that the entry angle γ_E is a function of both the retrorocket alignment angle β and the impulsive velocity decrement ΔV. Therefore, for constant γ_E,

$$\left.\frac{\partial \Delta V}{\partial \beta}\right|_{\gamma_E} = -\left.\frac{\partial \gamma_E}{\partial \beta}\right|_{\Delta V} \div \left.\frac{\partial \gamma_E}{\partial \Delta V}\right|_{\beta} \tag{5}$$

If the alignment angle β is found which maximizes γ_E ($\partial\gamma_E/\partial\beta|_{\Delta V} = 0$), then the minimum ΔV for a given entry angle also has been found ($\partial\Delta V/\partial\beta|_{\gamma_E} = 0$) provided that $\partial\gamma_E/\partial\Delta V|_\beta \neq 0$

As shown by Galman,[3] $\partial\gamma_E/\partial\beta|_{\Delta V}$ is given by

$$\left.\frac{\partial \gamma_E}{\partial \beta}\right|_{\Delta V} = \frac{\Delta V \sin\beta}{V_E^2(V_{c_1} - \Delta V\cos\beta)} \times \left[V_{c_1}\Delta V \cos\beta - (\Delta V)^2 + 2V_{c_1}^2\left(1 - \frac{r_1}{r_E}\right)\right]\frac{1}{\tan\gamma_E} \tag{6}$$

with the zero roots at

$$\sin\beta = 0 \tag{7a}$$

$$\cos\beta = \Delta V/V_{c_1} + \frac{2[(r_1/r_E) - 1]}{\Delta V/V_{c_1}} \tag{7b}$$

By using Eqs. (1–4), $\partial\gamma_E/\partial\Delta V|_\beta$ is given by

$$\left.\frac{\partial \gamma_E}{\partial \Delta V}\right|_{\beta} = \frac{\left\{\dfrac{V_{c_1}\Delta V \sin^2\beta - 2V_{c_1}^2[1 - (r_1/r_E)]\cos\beta}{V_E^2(V_{c_1} - \Delta V\cos\beta)}\right\}}{\tan\gamma_E} \tag{8}$$

But this expression can be zero only when

$$\cos\beta = \frac{(r_1/r_E) - 1}{\Delta V/V_{c_1}} \pm \left\{1 + \left[\frac{(r_1/r_E) - 1}{\Delta V/V_{c_1}}\right]^2\right\}^{1/2} \tag{9}$$

It is evident that Eqs. (7b) and (9) never can be equal for nonzero values of ΔV. Therefore, the β given by Eq. (7) will yield the maximum value of γ_E and the minimum value of ΔV for a fixed orbital altitude. For $\beta = 0$, the minimum value of ΔV is obtained by using Eqs. (1–4) and (7a) to eliminate V_E, V_1, γ_1, and β:

$$\left.\frac{\Delta V}{V_{c_E}}\right)_{\beta=0} = \left(\frac{r_E}{r_1}\right)^{1/2} \times \left\{1 - \frac{r_E}{r_1}\cos\gamma_E\left[\frac{2[(r_1/r_E) - 1]}{1 - [(r_E/r_1)\cos\gamma_E]^2}\right]^{1/2}\right\} \tag{10}$$

For $\beta \neq 0$, the minimum ΔV is given by

$$\left.\frac{\Delta V}{V_{c_E}}\right)_{\min} = \left\{\frac{r_E}{r_1}\left[1 - \left(\frac{r_E}{r_1}\cos\gamma_E\right)^2 - 2\left(\frac{r_1}{r_E} - 1\right)\right]\right\}^{1/2} \tag{11}$$

The optimum retrorocket alignment angle is found by combining Eqs. (7b) and (11):

$$\cos\beta)_{\text{opt}} = \frac{1 - (r_E/r_1\cos\gamma_E)^2}{\{1 - (r_E/r_1\cos\gamma_E)^2 - 2[(r_1/r_E) - 1]\}^{1/2}} \tag{12}$$

The velocity decrements and the alignment angle given by Eqs. (10–12) are shown in Fig. 2§ for various values of atmospheric entry angle and radius ratio r_1/r_0. For a given γ_E and for small radius ratios, Fig. 2 shows that it is advantageous to operate at the optimum alignment angle given by Eq. (12). As the radius ratio increases, the alignment angle decreases

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